

# Fibonacci Numbers and Binet's Formula

## Roadmap (Überblick)

### ■ The generalized Fibonacci equation

#### ■ 0. A remark on the historical development

In his *Liber abaci* (1202), Leonardo of Pisa (called *Fibonacci*, ca. 1170 - 1250) formulated a problem giving rise to the following famous sequence of numbers now called the "Fibonacci" numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Its most important property is that every member of the sequence is the sum of its two immediate predecessors (except for the initial values):

$$F_{k+2} = F_{k+1} + F_k$$

It took several centuries until J. P. M. Binet (1786-1856) finally presented the following formula for the Fibonacci numbers:

$$F_k = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^k$$

#### ■ 1. Generalisation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0 \quad (* \text{ GFE } *)$$

This equation clearly generalizes the Fibonacci equation; we will, therefore, call it the *generalized Fibonacci equation* (GFE) for short.

#### ■ 2. Applying standard methodology: Introducing new parameters (in this case: a "tuning" parameter)

Due to this result we have to handle the decomposition in a slightly more subtle way by introducing an extra parameter called  $t$  (for "tuning") in the following way.

$$y_{k+2} + (a_1 + t) \cdot y_{k+1} = 0$$

$$-t \cdot y_{k+1} + a_0 \cdot y_k = 0$$

GFE can be thought of as being the sum of these first-order equations. If, by choosing a suitable "tuning" value for  $t$ , we can make these two first-order equations identical, then they will have the same closed form representations and we can try to combine their individual solutions into a solution for the generalized Fibonacci equation.

### ■ 3. The tuning process

Written in the "standard" form for geometric sequences the last two first-order equations read

$$y_{k+2} = -(a_1 + t) \cdot y_{k+1} \quad (* \text{ GS-1 } *)$$

$$y_{k+1} = \frac{a_0}{t} \cdot y_k \quad (* \text{ GS-2 } *)$$

These difference equations for geometric sequences are identical if their coefficients  $-(a_1 + t)$  and  $\frac{a_0}{t}$  are equal. (The "index-shift" by 1 is irrelevant, since the equations are valid for all values of  $k$ ).

A necessary condition for equality, hence, is

$$-(a_1 + t) = \frac{a_0}{t}$$

### ■ 4. The characteristic polynomial

$$t^2 + a_1 \cdot t + a_0 = 0$$

Thus, the above geometric sequences are identical if the "tuning" parameter  $t$  satisfies the so-called *characteristic equation* of (\* GFE \*):

$$x^2 + a_1 \cdot x + a_0 = 0$$

We finally obtain the tuning parameters

$$t_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}$$

$$t_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}$$

### ■ 5. Solutions - by applying the results on geometric series

Thus, the geometric sequences adding up to (\* GFE \*) are

1. By using the root  $t_1$ :

$$y_{k+2} = -\left(a_1 + \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}} \cdot y_k$$

2. By using the root  $t_2$ :

$$y_{k+2} = -\left(a_1 + \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}\right) \cdot y_{k+1}$$

$$y_{k+1} = \frac{a_0}{\frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}} \cdot y_k$$

## ■ 6. Vieta's formulae

The tuning parameters  $t_1$  and  $t_2$ , being the roots of the quadratic equation

$$x^2 + a_1 \cdot x + a_0 = 0$$

must satisfy Vieta's equations:

$$t_1 + t_2 = -a_1 \quad \text{and} \quad t_1 \cdot t_2 = a_0$$

Thus, by substituting either  $t_1$  or  $t_2$  for  $t$  and applying Vieta's formulae we can rewrite (\* GS-1 \*) and (GS-2 \*) in the following way:

$$y_{k+2} = t_1 \cdot y_{k+1} \quad (* \text{ GS-1.1 } *)$$

$$y_{k+2} = t_2 \cdot y_{k+1} \quad (* \text{ GS-1.2 } *)$$

$$y_{k+1} = t_1 \cdot y_k \quad (* \text{ GS-2.1 } *)$$

$$y_{k+1} = t_2 \cdot y_k \quad (* \text{ GS-2.2 } *)$$

By pure combinatorics, substituting two roots into two equations formally gives four cases, but by algebra (Vieta) these melt down to two essentially different cases.

## ■ 7. Results

### **Theorem (solutions of the generalized Fibonacci equation)**

(cf. [DZ 1989], Satz 13.1, page 90)

The generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

has the following "solutions" (i.e. closed form representations):

$$(a1) \quad y_k = \left( \frac{-a_1 + \sqrt{a_1^2 - 4 a_0}}{2} \right)^k$$

and

$$(a2) \quad y_k = \left( \frac{-a_1 - \sqrt{a_1^2 - 4 a_0}}{2} \right)^k$$

## ■ 8. Obtaining more solutions

### Theorem (combining solutions)

If the sequences

$$(u_k)_{k=0,\dots,\infty} \quad \text{and} \quad (v_k)_{k=0,\dots,\infty}$$

are solutions of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

then their "sum"

$$(a1) \quad (u_k)_{k=0,\dots,\infty} \oplus (v_k)_{k=0,\dots,\infty} := (u_k + v_k)_{k=0,\dots,\infty}$$

and for any real number  $C$  the "scalar multiple"

$$(a2) \quad C \odot (u_k)_{k=0,\dots,\infty} := (C \cdot u_k)_{k=0,\dots,\infty}$$

also are solutions of the generalized Fibonacci equation.

Henceforth we will use the simpler operation symbols  $+$  and  $\cdot$  instead of  $\oplus$  and  $\odot$ .

### Corollary (linear combination of solutions):

If the sequences

$$(u_k)_{k=0,\dots,\infty} \quad \text{and} \quad (v_k)_{k=0,\dots,\infty}$$

are solutions of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

then for any real numbers  $C_1$  and  $C_2$  the "linear combination"

$$C_1 \cdot (u_k)_{k=0,\dots,\infty} + C_2 \cdot (v_k)_{k=0,\dots,\infty}$$

also are solutions of the generalized Fibonacci equation.

### Corollary

The generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

has the following "solutions" (i.e. closed form representations):

$$C_1 \cdot \left( \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \right)^k + C_2 \cdot \left( \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \right)^k \quad (* \text{ S-GFE } *)$$

where  $C_1$  and  $C_2$  are arbitrary real (or complex) numbers.

In other words: The set of all solutions of the generalized Fibonacci equation is a *vector space* over a suitable scalar field (usually the field the coefficients are taken from); cf. [DZ 1989], Satz 14.1, page 91.

Furthermore, it is not difficult to see that the dimension of this vector space is 2 and that, in case the solutions  $t_1$  and  $t_2$  of GFE's characteristic equation do not coincide, then the sequences  $(t_1)^k$  and  $(t_2)^k$  are a basis of this vector space.

## ■ 9. Initial values

The above results were valid independent of any initial values  $y_0$  and  $y_1$  of the GFE. Let us now assume that the solution (\* S-GFE \*), additionally, is to satisfy these initial values. Then for  $k = 0$  and  $k = 1$  the following two linear equations will have to be satisfied by  $C_1$  and  $C_2$ :

### Corollary

The initial values of the generalized Fibonacci equation

$$y_{k+2} + a_1 \cdot y_{k+1} + a_0 \cdot y_k = 0$$

can be expressed in the following way:

$$C_1 \cdot \left( \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \right)^0 + C_2 \cdot \left( \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \right)^0 = y_0 \quad (* \text{ LE-1 } *)$$

$$C_1 \cdot \left( \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \right)^1 + C_2 \cdot \left( \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \right)^1 = y_1 \quad (* \text{ LE-2 } *)$$

i.e.

$$C_1 + C_2 = y_0$$

$$C_1 \cdot \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} + C_2 \cdot \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} = y_1$$

Notwithstanding the algebraic complexity of these equations, they are two simple linear equations in the two unknowns  $C_1$  and  $C_2$  which can be solved by straightforward algebraic procedures.

**Exercise:** Show that

$$C_1 = - \frac{-a_1 y_0 - \sqrt{-4 a_0 + a_1^2} y_0 - 2 y_1}{2 \sqrt{-4 a_0 + a_1^2}}$$

and

$$C_2 = - \frac{a_1 y_0 - \sqrt{-4 a_0 + a_1^2} y_0 + 2 y_1}{2 \sqrt{-4 a_0 + a_1^2}}$$

are solutions of (\* LE-1 \*) and (\* LE-2 \*).

## ■ 10. Applying the results to the sequence of the standard Fibonacci numbers - Binet's formula

The sequence of the standard Fibonacci numbers, equivalently either starting with index 0 or index 1 is given by

$$\begin{array}{cccccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & \dots \\ 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \dots \end{array}$$

Specializing from GFE, its parameters are:

$$a_1 = -1 \quad \text{and} \quad a_0 = -1.$$

Hence, the homogeneous equation

$$y_{k+2} - y_{k+1} - y_k = 0$$

has the "general" solution

$$y_k = C_1 \cdot \left( \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \right)^k + C_2 \cdot \left( \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \right)^k$$

i.e.

$$y_k = C_1 \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^k + C_2 \cdot \left( \frac{1 - \sqrt{5}}{2} \right)^k$$

$$\left\{ \left\{ C_1 \rightarrow -\frac{-a_1 y_0 - \sqrt{-4a_0 + a_1^2} y_0 - 2y_1}{2\sqrt{-4a_0 + a_1^2}}, C_2 \rightarrow -\frac{a_1 y_0 - \sqrt{-4a_0 + a_1^2} y_0 + 2y_1}{2\sqrt{-4a_0 + a_1^2}} \right\} \right\}$$

$$C_1 = -\frac{-a_1 y_0 - \sqrt{-4a_0 + a_1^2} y_0 - 2y_1}{2\sqrt{-4a_0 + a_1^2}} \quad /. \{a_0 \rightarrow -1, a_1 \rightarrow -1, y_0 \rightarrow 0, y_1 \rightarrow 1\}$$

$$\frac{1}{\sqrt{5}}$$

$$C_2 = -\frac{a_1 y_0 - \sqrt{-4a_0 + a_1^2} y_0 + 2y_1}{2\sqrt{-4a_0 + a_1^2}} \quad /. \{a_0 \rightarrow -1, a_1 \rightarrow -1, y_0 \rightarrow 0, y_1 \rightarrow 1\}$$

$$-\frac{1}{\sqrt{5}}$$

## ■ 11. Binet's Formula

### Theorem (Binet)

The Fibonacci equation

$$y_{k+2} = y_{k+1} + y_k$$

has the following "solution" (i.e. closed form representations):

$$y_k = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^k$$

## ■ 12. Check

$$\text{Table}[\text{Simplify}\left[\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^k\right], \{k, 0, 30\}]$$

{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040}